

THE QUASI-ORBIT SPACE OF CONTINUOUS C^* -DYNAMICAL SYSTEMS

BY

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ABSTRACT. Let (A, G, α) be a separable continuous C^* -dynamical system. Suppose G is amenable and α is free on the dual \hat{A} of A . Then the quasi-orbit space $(\text{Prim } A/\alpha)^\sim$ of the primitive ideal space $\text{Prim } A$ of A by α is homeomorphic to the induced primitive ideal space which is dense in the primitive ideal space $\text{Prim } C^*(A; \alpha)$ of the C^* -crossed product $C^*(A; \alpha)$ of A by α .

1. Introduction. The theory of crossed products of operator algebras has been developed by a number of people since von Neumann constructed the examples of factors. Among its various studies, Effros-Hahn [3] topologically characterized under certain conditions the primitive ideal space of C^* -crossed products as the quasi-orbit space of transformation groups for separable abelian discrete C^* -dynamical systems. Succeedingly, Zeller-Meier [7] generalized their result in separable discrete C^* -dynamical systems. However, several different aspects come out in the continuous case. For instance, the original C^* -algebra is never imbedded in its crossed product as a sub C^* -algebra, therefore it has no associated conditional expectations from the crossed product. Moreover, it is unclear that any element in crossed products can be described as a Fourier expansion in certain fashion.

In this paper, we shall discuss the quasi-orbit space of separable continuous C^* -dynamical systems, specifically, given a separable continuous C^* -dynamical system (A, G, α) . Suppose G is amenable and α is free on the dual \hat{A} of A . Then the quasi-orbit space $(\text{Prim } A/\alpha)^\sim$ of the primitive ideal space $\text{Prim } A$ of A by α is homeomorphic to the induced primitive ideal space $\{(\text{Ind } \rho)^{-1}(0)\}_{\rho \in \hat{A}}$ which is dense in the primitive ideal space $\text{Prim } C^*(A, \alpha)$ of the C^* -crossed product $C^*(A, \alpha)$ of A by α . This can be considered as a generalization of Zeller-Meier's result.

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2. Preliminaries. According to Effros-Hahn [3], Dang-Ngoc-Guichardet [2] Takai [5], [6], and Zeller-Meier [7], we shall briefly review several basic notions and fundamental results which will be used later. A triple (A, G, α) of a C^* -algebra A , a locally compact group G , and a $*$ -homomorphism α of G into $\text{Aut}(A)$ is called a *continuous C^* -dynamical system* if α is pointwise norm continuous, where $\text{Aut}(A)$ is the set of all $*$ -automorphisms of A . It is said to be *separable* if A and G are separable. Moreover, the action α is said to be *free* on a subset S of the representation space $\text{Rep } A$ of A if given a $g \neq e \in G$, and a $\rho \neq 0 \in S$, any subrepresentation $\rho' \neq 0$ of ρ dominates a representation $\rho'' \neq 0$ with $\alpha_g \cdot \rho'' \not\leq \rho''$, where $(\alpha_g \cdot \rho'')(a) = \rho'' \circ \alpha_g^{-1}(a)$ for $a \in A, g \in G$. Let $\text{Fac } A$ be the set of all factor representations of A . Since any nonzero subrepresentation of ρ is quasi-equivalent to ρ if $\rho \in \text{Fac } A$, freeness of α on S for a subset S of $\text{Fac } A$ is equivalent to saying that $\alpha_g \cdot \rho \not\leq \rho$ for every $\rho \neq 0 \in S$. Furthermore, due to the fact that freeness is congruent with quasi-equivalence (resp. equivalence), it can be defined on the quasi-dual \hat{A} (resp. the dual \hat{A}) of A .

Now let (A, G, α) be a continuous C^* -dynamical system. According to Effros-Hahn [3], and Zeller-Meier [7], the quasi-orbit space $(\text{Prim } A/\alpha)^\sim$ of the primitive ideal space $\text{Prim } A$ of A by α is defined by the set of all orbit closures in $\text{Prim } A$. It can be identified with the quotient space of $\text{Prim } A$ by the following equivalence relation: $p \sim q$ if and only if $(G \cdot p)^- = (G \cdot q)^-$, where $(G \cdot p)^-$ is the closure of the orbit of $p \in \text{Prim } A$ by α . Then, it is easily seen that $(\text{Prim } A/\alpha)^\sim$ is a T_0 -space with respect to the quotient topology of $\text{Prim } A$. Define a Borel structure on $(\text{Prim } A/\alpha)^\sim$ by the quotient map of $\text{Prim } A$. Then, it is saturated with the quotient topology. Suppose A is separable, each element of $(\text{Prim } A/\alpha)^\sim$ is Borel since $(\text{Prim } A/\alpha)^\sim$ is a T_0 -space (cf. [3]).

The following lemma has the key role in finding a mapping from the primitive ideal space $\text{Prim } C^*(A, \alpha)$ of the crossed product $C^*(A; \alpha)$ into $(\text{Prim } A/\alpha)^\sim$:

LEMMA 2.1. *Let (A, G, α) be a separable continuous C^* -dynamical system. Suppose there exists a projection valued measure on $(\text{Prim } A/\alpha)^\sim$ whose values are zero or identity operators. Then it is concentrated in one point (cf. [3]). Notice that it is unnecessary to assume the separability of G to show the above lemma.*

Next, we shall define the C^* -crossed product (resp. the reduced C^* -crossed product) $C^*(A; \alpha)$ (resp. $C_r^*(A; \alpha)$) of C^* -algebra A by a continuous action α of a locally compact group as follows: Let $L^1(A; G)$ be the set of all A -valued Bochner integrable measurable functions on G with respect to the left Haar measure

α_g of G with the $*$ -algebraic structure given by

$$(xy)(g) = \int_G x(h) \alpha_h[y(h^{-1}g)] dh$$

and

$$x^*(g) = \Delta(g)^{-1} \alpha_g[x(g^{-1})]^*,$$

where Δ is the modular function of G . Then, $C^*(A; \alpha)$ (resp. $C_r^*(A; \alpha)$) is the completion of $L^1(A; \alpha)$ with respect to the C^* -norm $\|\cdot\|$ (resp. $\|\cdot\|_r$) defined by

$$\|x\| = \sup\{\|\pi(x)\|: \pi \in \text{Rep } L^1(A; G)\}$$

and

$$\|x\|_r = \sup\{\|(\text{Ind } \rho)(x)\|: \rho \in \text{Rep } A\},$$

where $\text{Ind } \rho$ is the representation of $C^*(A; \alpha)$ induced by ρ (cf. [5], [6]). It is clear that $C_r^*(A; \alpha)$ is a quotient algebra of $C^*(A; \alpha)$. Let us define a positive linear functional $\tilde{\varphi}_f$ on $C^*(A; \alpha)$ determined by

$$\tilde{\varphi}_f(x) = \iint_{G \times G} f(g^{-1}h) \overline{f(h)} \varphi \circ \alpha_h^{-1}[x(g)] dg dh$$

for $x \in L^1(A; G)$, where φ is in the set $(A^*)_+$ of all positive linear functionals on A , and f is in the $*$ -algebra $K(G)$ of all continuous functions on G with compact support. Then, one can estimate $\|(\text{Ind } \rho)(x)\|$ by $\tilde{\varphi}_f$ as follows:

$$(2.1) \quad \begin{aligned} & \|(\text{Ind } \rho)(x)\| \\ &= \sup \left\{ \frac{\tilde{\varphi}_f(y^* x^* xy)^{1/2}}{\tilde{\varphi}_f(y^* y)^{1/2}} : y \in L^1(A; G), f \in K(G), \varphi \in \Omega(y; f) \cap w(\rho) \right\} \end{aligned}$$

where $\Omega(y; f) = \{\varphi \in (A^*)_+ : \tilde{\varphi}_f(y^* y) \neq 0\}$, and $w(\rho) = \{\varphi \in (A^*)_+ : \pi_\varphi \in \text{Rep } A \text{ is weakly contained in } \rho\}$ (cf. [5]).

We now state a sufficient condition under which $C^*(A; \alpha)$ is equal to $C_r^*(A; \alpha)$.

LEMMA 2.2. *Let (A, G, α) be a continuous C^* -dynamical system. Suppose G is amenable as a topological group, then we have*

$$C^*(A; \alpha) = C_r^*(A; \alpha) \quad (\text{cf. [5], [6]}).$$

Moreover, we present a condition under which $\text{Ind } \rho$ is faithful on $C_r^*(A; \alpha)$.

LEMMA 2.3. *Let (A, G, α) be as in the preceding, and ρ in $\text{Rep } A$. Then the following conditions are equivalent:*

- (I) $\sum_{g \in G}^{\oplus} \alpha_g \cdot \rho$ is faithful on A ,
- (II) $\text{Ind } \rho$ is faithful on $C_r^*(A; \alpha)$ (cf. [5]).

By the above lemma, one knows without difficulty that faithfulness of ρ implies that of $\text{Ind } \rho$.

Finally, we shall write down a couple of special notations for later use. Let (A, G, α) be a continuous C^* -dynamical system. Then given $X \in C^*(A; \alpha)$, $a \in A$, there exists an element $aX \in C^*(A; \alpha)$, with $(aX)(g) = aX(g)$ if $X \in L^1(A; G)$. Let I be an ideal of $C^*(A; \alpha)$ where "ideal" involves closed and two-sided throughout this paper. Denote by $[A : I]$ the set of all elements a in A such that $aX \in I$ for all $X \in C^*(A; \alpha)$. It is not evident whether $[A : I]$ is an ideal of A . However, the following observation tells us that it is affirmative:

LEMMA 2.4. *Let (A, G, α) be a continuous C^* -dynamical system, and $\pi \in \text{Rep } C^*(A; \alpha)$. Suppose ρ is in $\text{Cov-rep } A$ corresponding to π , then we have*

$$[A : \pi^{-1}(0)] = \rho^{-1}(0) \quad (\text{cf. [3]})$$

where $\text{Cov-rep } A$ is the set of all covariant representations of A , and $\pi^{-1}(0)$ (resp. $\rho^{-1}(0)$) is the kernel of π (resp. ρ). From this fact, $[A : I]$ for an ideal I of $C^*(A; \alpha)$ is an invariant ideal of A under the action α . (It is called α -invariant.)

3. The quasi-orbit space of continuous C^* -dynamical systems. In this section, we shall discuss the topological relation between the quasi-orbit space of continuous C^* -dynamical systems and the primitive ideal space of C^* -crossed products under certain conditions. Let (A, G, α) be a separable continuous C^* -dynamical system. Given a π in $\text{Fac } C^*(A; \alpha)$, there exists a unique ρ in $\text{Cov-rep } A$ corresponding to π . Due to Glimm [4], there exists a unique projection valued measure μ_ρ on the Borel structure of $\text{Prim } A$ such that

$$(3.1) \quad \mu_\rho(\bar{U}) = \text{Proj}[\rho(k(\bar{U}^c))\mathfrak{h}_\rho]$$

for every open set \bar{U} in $\text{Prim } A$, where Proj means projection and \bar{U}^c is the complement of \bar{U} in $\text{Prim } A$, and $k(\bar{U}^c)$ is the kernel of \bar{U}^c . Moreover, the range of μ_ρ is contained in the center z_ρ of the von Neumann algebra $\rho(A)^-$ generated by $\rho(A)$. Since π is primary and ρ is covariant, the measure μ_ρ is ergodic. Namely, the values of α -invariant Borel sets under μ_ρ are zero or identity operators on \mathfrak{h}_ρ . Let η be the natural mapping from $\text{Prim } A$ onto $(\text{Prim } A/\alpha)^\sim$, and let ν_ρ be the image of μ_ρ by η . Then the range of ν_ρ is zero or identity operators on \mathfrak{h}_ρ . By Lemma 2.1, there exists an element $\eta(p)$ in $(\text{Prim } A/\alpha)^\sim$ at which ν_ρ is concentrated. We claim that $(G \cdot p)^-$ is the support $\text{Supp } \mu_\rho$ of μ_ρ . In fact, since $\eta^{-1} \circ \eta(p) \subset (G \cdot p)^-$, it follows that $\text{Supp } \mu_\rho \subset (G \cdot p)^-$. By the fact that $\text{Supp } \mu_\rho$ is α -invariant closed in $\text{Prim } A$, it contains $(G \cdot p)^-$, or must be disjoint from $\eta^{-1} \circ \eta(p)$. However, the latter case does not occur since μ_ρ is concentrated in $\eta^{-1} \circ \eta(p)$.

Consequently, one gets $\text{Supp } \mu_\rho = (G \cdot p)^-$. Moreover, it can be verified that $\text{Supp } \mu_\rho = h(\rho^{-1}(0))$ where $h(\rho^{-1}(0))$ is the hull of $\rho^{-1}(0)$. Actually, it is clear by definition that $h(\rho^{-1}(0))^c$ is an open null set with respect to μ_ρ . Therefore, $\text{Supp } \mu_\rho \subset h(\rho^{-1}(0))$. Suppose $\text{Supp } \mu_\rho$ is not equal to $h(\rho^{-1}(0))$. Then, there exists an open neighborhood \bar{U} of an element q in $h(\rho^{-1}(0))$ from which $\text{Supp } \mu_\rho$ is disjoint. Hence, it follows that $k(\bar{U}^c) \subset \rho^{-1}(0)$. Taking hull, one obtains that \bar{U} is disjoint from $h(\rho^{-1}(0))$, which is a contradiction. Summing up the argument discussed above, we have the following lemma:

LEMMA 3.1. *Let (A, G, α) be a separable continuous C^* -dynamical system, and let π be in $\text{Fac } C^*(A; \alpha)$. Suppose ρ is in $\text{Cov-rep } A$ corresponding to π . Then there exist a unique (up to equivalence) projection valued Borel measure μ_ρ on $\text{Prim } A$, and an element P in $\text{Prim } A$ such that*

$$\text{Supp } \mu_\rho = (G \cdot p)^- = h(\rho^{-1}(0)).$$

REMARK. The measure μ_ρ also can be realized in the following way: Let $\rho = \int_{\hat{A}}^{\oplus} \rho_{\hat{s}} d\nu(\hat{s})$ be the central decomposition of ρ . Using the canonical mapping ϕ from \hat{A} onto $\text{Prim } A$, one gets the image μ of ν under ϕ . It is nothing but μ_ρ up to equivalence.

By Lemmas 2.4 and 3.1, we can define a mapping Φ of $\text{Prim } C^*(A; \alpha)$ into $(\text{Prim } A/\alpha)^\sim$ by

$$(3.2) \quad \Phi(\mathfrak{P}) = h([A : \mathfrak{P}]).$$

Then, we have the following:

LEMMA 3.2. *The mapping Φ defined above is continuous from $\text{Prim } C^*(A; \alpha)$ with the hull kernel topology into $(\text{Prim } A/\alpha)^\sim$ with the quotient topology.*

PROOF. Though the proof goes on as in the abelian case, we shall trace it for completeness. Given a closed set F in $(\text{Prim } A/\alpha)^\sim$, it suffices to show that there exists a subset I of $C^*(A; \alpha)$ such that $\Phi^{-1}(F) = h(I)$.

Define a set I by $\{aX : a \in k(F_1), X \in C^*(A; \alpha)\}$ where $F_1 = \eta^{-1}(F)$. Then, this set enjoys the property cited above. In fact, for any $\mathfrak{P} \in \Phi^{-1}(F)$, there exists a π in $\text{Irr } C^*(A; \alpha)$ whose kernel is \mathfrak{P} . Let ρ be in $\text{Cov-rep } A$ corresponding to π . By Lemma 3.1, there exists a P in $\text{Prim } A$ such that $h([A : \mathfrak{P}]) = (G \cdot p)^-$. Since $\Phi(\mathfrak{P}) \in F$, it follows by (3.2) that $p \in \eta^{-1}(F) = F_1$. Since F_1 is α -invariant closed in $\text{Prim } A$, it contains $(Gp)^-$, which means that $h([A : \mathfrak{P}]) = F_1$. Taking the kernel, one gets that $k(F_1) \subset [A : \mathfrak{P}] = \rho^{-1}(0)$. Since $\pi(ax) = \rho(a)\pi(x)$ for $a \in A$ and $X \in C^*(A; \alpha)$, it is verified that $I \subset \pi^{-1}(0) = \mathfrak{P}$. Thus, one obtains that $\mathfrak{P} \in h(I)$. Since one can trace the above argument back until its starting point, one gets the conclusion. Q.E.D.

From now on, let us assume that the action α is free on \hat{A} . Then, the next lemma guarantees that Φ is surjective:

LEMMA 3.3. *Suppose α is free on \hat{A} . Then, one has for every $\rho \in \text{Irr } A$,*

- (i) $\text{Ind } \rho \in \text{Irr } C^*(A; \alpha)$, and
- (ii) $\Phi[(\text{Ind } \rho)^{-1}(0)] = (G \cdot \rho^{-1}(0))^-$,

where Irr means the set of all irreducible representations.

PROOF. (i) Let $G_{\hat{\rho}}$ be the stabilizer of the equivalence class $\hat{\rho} \in \hat{A}$ of ρ , and let $(\bar{\rho}, \lambda)$ be in $\text{Cov-rep}(A, G)$ corresponding to $\text{Ind } \rho$. Let \bar{U} be a strongly measurable cocycle representation of $G_{\hat{\rho}}$ such that $\bar{U}_g^* \rho(a) \bar{U}_g = \alpha_g \cdot \rho(a)$ for $a \in A$, $g \in G_{\hat{\rho}}$. Define $\tilde{U}_g, g \in G_{\hat{\rho}}$, by

$$(\tilde{U}_g \xi)(h) = \Delta(\rho)^{1/2} \bar{U}_g \xi(hg) \quad \text{for } \xi \in L^2(\mathfrak{h}_{\hat{\rho}}; G), h \in G.$$

Due to Busby-Smith [1], it can be shown that the commutant $\bar{\rho}(A)'$ of $\bar{\rho}$ is generated by the $\tilde{U}_g, g \in G_{\hat{\rho}}$, and the diagonalizable algebra. Since α is free on \hat{A} , it follows that $\bar{\rho}(A)'$ is the diagonal algebra. Since the only diagonals which commute with λ are the constant, one gets the statement (i).

(ii) By definition, it suffices to show that $h(\bar{\rho}^{-1}(0)) = (G \cdot \rho^{-1}(0))^-$. Since $\bar{\rho}^{-1}(0) = \bigcap_{g \in G} (\alpha_g \cdot \rho)^{-1}(0)$, it follows that

$$h(\bar{\rho}^{-1}(0)) = h\left(\bigcap_{g \in G} (\alpha_g \cdot \rho)^{-1}(0)\right) \supset \left(\bigcup_{g \in G} h[(\alpha_g \cdot \rho)^{-1}(0)]\right)^-$$

which implies that $(G \cdot \rho^{-1}(0))^- \subset h(\bar{\rho}^{-1}(0))$. On the other hand, let $\mu_{\bar{\rho}}$ be the Borel measure determined by $\bar{\rho}$ as (3.1). Then one gets by simple computation that $\text{Supp } \mu_{\bar{\rho}} \subset (G \cdot \rho^{-1}(0))^-$. Thus, the statement (ii) follows by Lemma 3.1. Q.E.D.

In general, it is not clear whether the mapping Φ is injective. However, one can get a partial answer under the amenability of G . Let E be the set of all $(\text{Ind } \rho)^{-1}(0)$, $\rho \in \text{Irr } A$. Then the following lemma tells us that Φ is injective on E :

LEMMA 3.4. *Let (A, G, α) be a separable continuous C^* -dynamical system, and let α be free on \hat{A} . Suppose G is amenable as a topological group. Then one has for every $\rho \in \text{Irr } A$,*

$$(\text{Ind } \rho)^{-1}(0) = C^*(\bar{\rho}^{-1}(0); \alpha)$$

where $\bar{\rho} \in \text{Cov-rep } A$ corresponding to $\text{Ind } \rho$.

PROOF. By definition, one easily gets that $(\text{Ind } \rho)^{-1}(0) \supset C^*(\bar{\rho}^{-1}(0); \alpha)$. To show the converse inclusion, one first observes that $C^*(\bar{\rho}^{-1}(0); \alpha)$ is a primitive ideal of $C^*(A; \alpha)$. In fact, one estimates by Lemmas 2.2 and 2.3 that

$$\begin{aligned}
C^*(A; \alpha)/(\text{Ind } \rho)^{-1}(0) &\cong (\text{Ind } \rho)[C^*(A; \alpha)] \\
&\cong (\text{Ind } \iota)[C_r^*(\bar{\rho}(A); \bar{\alpha})] \cong C_r^*(\bar{\rho}(A); \bar{\alpha}) = C^*(\bar{\rho}(A); \bar{\alpha}) \\
&\cong C^*(A/\bar{\rho}^{-1}(0); \tilde{\alpha}) \cong C^*(A; \alpha)/C^*(\bar{\rho}^{-1}(0); \alpha),
\end{aligned}$$

where $\bar{\alpha}$ (resp. $\tilde{\alpha}$) is the continuous action of G on $\bar{\rho}(A)$ (resp. $A/\bar{\rho}^{-1}(0)$) determined by $\bar{\rho}$ (resp. the natural map of A onto $A/\bar{\rho}^{-1}(0)$), and ι is the trivial automorphism of $\bar{\rho}(A)$. Therefore, there exists a π in $\text{Irr } C^*(A; \alpha)$ with $\pi^{-1}(0) = C^*(\bar{\rho}^{-1}(0); \alpha)$. Since every nonzero vector ξ in \mathfrak{h}_π is cyclic for π , it suffices to show that $\pi_\xi^{-1}(0) \supset (\text{Ind } \rho)^{-1}(0)$ for π_ξ corresponding to $\omega_\xi \circ \pi$. Let (L, ∇) be in $\text{Cov-rep}(A, G)$ associated with π . Then, one can see by Lemma 2.4 and $\pi^{-1}(0) = C^*(\bar{\rho}^{-1}(0); \alpha)$ that

$$(3.3) \quad L^{-1}(0) = [A : \pi^{-1}(0)] = \bar{\rho}^{-1}(0).$$

Since $\bar{\rho}^{-1}(0) = (\sum_{g \in G}^\oplus \alpha_g \cdot \rho)^{-1}(0)$, the equality (3.3) means that L is weakly equivalent to $\sum_{g \in G}^\oplus \alpha_g \cdot \rho$. On the other hand, the amenability of G gives us a sequence $(f_n)_n$ of $K(G)$ such that $\tilde{f}_n * f_n$ converges to 1 with respect to compact open topology where $\tilde{f}_n(g) = \overline{f_n(g^{-1})}$. Let us take a positive linear functional Ψ_n on $C^*(A; \alpha)$ determined by $\Psi_n(x) = \int_G (\tilde{f}_n * f_n)(g) \langle L[x(g)] \nabla(g) \xi | \xi \rangle dg$ for $x \in L^1(A; G)$. Then, it can be shown by [5] that

$$(3.4) \quad \Psi_n(y^* x^* x y) \leq \|y\|^2 \|\tilde{f}_n \otimes \xi\|^2 \|(\text{Ind } L)(x)\|^2$$

for every $x, y \in C^*(A; \alpha)$, and $n = 1, 2, \dots$. Since L is weakly equivalent to $\sum_{g \in G}^\oplus \alpha_g \cdot \rho$, it follows by (2.1) that

$$(3.5) \quad \|(\text{Ind } L)(x)\| = \|\text{Ind}(\sum_{g \in G}^\oplus \alpha_g \cdot \rho)(x)\|.$$

Since $\text{Ind}(\sum_{g \in G}^\oplus \alpha_g \cdot \rho)$ (resp. $\text{Ind } \alpha_g \cdot \rho$) is equivalent to $\sum_{g \in G}^\oplus \text{Ind } \alpha_g \cdot \rho$ (resp. $\text{Ind } \rho$), the equality (3.5) implies that

$$\begin{aligned}
\|(\text{Ind } L)(x)\| &= \|\sum_{g \in G}^\oplus (\text{Ind } \alpha_g \cdot \rho)(x)\| = \sup_{g \in G} \|(\text{Ind } \alpha_g \cdot \rho)(x)\| \\
&= \|(\text{Ind } \rho)(x)\|.
\end{aligned}$$

Thus, one obtains by (3.4) that

$$\Psi_n(y^* x^* x y) \leq \|y\|^2 \|\tilde{f}_n \otimes \xi\|^2 \|(\text{Ind } \rho)(x)\|^2$$

for every $x, y \in C^*(A; \alpha)$ and $n = 1, 2, \dots$. Now suppose x is in $(\text{Ind } \rho)^{-1}(0)$. Then, it follows that $\Psi_n(y^* x^* x y) = 0$ for all $y \in C^*(A; \alpha)$ and $n = 1, 2, \dots$.

Since Ψ_n converges to $w_\xi \circ \pi$ weakly, $w_\xi \circ \pi(y^*x^*xy) = 0$ for all $y \in C^*(A; \alpha)$, which means that $x \in \pi_\xi^{-1}(0)$. Q.E.D.

By the above lemma, the mapping Φ is bijective from E onto $(\text{Prim } A/\alpha)^\sim$. Combining all the lemmas obtained above, we shall show the following theorem which is a generalization of the result which is obtained by Effros-Hahn, and Zeller-Meier (cf. [3], [7]):

THEOREM 3.5. *Let (A, G, α) be a separable continuous C^* -dynamical system. Suppose G is amenable and α is free on the dual \hat{A} of A . Then, the quasi-orbit space $(\text{Prim } A/\alpha)^\sim$ of the primitive ideal space $\text{Prim } A$ of A by α is topologically imbedded in the primitive ideal space $\text{Prim } C^*(A; \alpha)$ of the C^* -crossed product $C^*(A; \alpha)$ of A by α .*

PROOF. Using the same notations denoted before, we shall show that Φ is a closed mapping from E onto $(\text{Prim } A/\alpha)^\sim$. Let F be a closed set in E . Define F_0 by $\eta^{-1}[\Phi(F)]$. It has to be shown that F_0 is closed in $\text{Prim } A$. Take a $p \in F_0$, and an $L \in \text{Irr } A$ with $L^{-1}(0) = p$. To prove $p \in F_0$, it suffices to show that

$$(3.6) \quad (\text{Ind } L)^{-1}(0) \supset \bigcap \{(\text{Ind } \rho)^{-1}(0) : \text{Ind } \rho \in F\}.$$

Since $p \in F_0^-$, it follows that

$$\begin{aligned} L^{-1}(0) &\supset \bigcap \{\eta^{-1}[(G \cdot \rho^{-1}(0))^-] : \text{Ind } \rho \in F\} \\ &\supset \bigcap \{(\alpha_g \cdot \rho)^{-1}(0) : \text{Ind } \rho \in F, g \in G\}, \end{aligned}$$

which implies that L is weakly contained in the set $\{\alpha_g \cdot \rho : \text{Ind } \rho \in F, g \in G\}$. By the similar method used in the previous lemma, one has that

$$\begin{aligned} \|(\text{Ind } L)(x)\| &\leq \sup \{\|(\text{Ind } \alpha_g \cdot \rho)(x)\| : \text{Ind } \rho \in F, g \in G\} \\ &= \sup \{\|(\text{Ind } \rho)(x)\| : \text{Ind } \rho \in F\} \end{aligned}$$

for all $x \in C^*(A; \alpha)$. Therefore, the inclusion (3.8) holds. Thus, the mapping Φ is a homeomorphism from E onto $(\text{Prim } A/\alpha)^\sim$. Q.E.D.

Since $k(E) = (0)$, it follows by the above theorem that $(\text{Prim } A/\alpha)^\sim$ is dense in $\text{Prim } C^*(A; \alpha)$. Moreover, it is true that $\text{Prim } C^*(A; \alpha) = \bigcup \{\mathfrak{P}^- : \mathfrak{P} \in E\}$. In fact, let $\pi^{-1}(0) \in \text{Prim } C^*(A; \alpha)$ where $\pi \in \text{Irr } C^*(A; \alpha)$. Taking $L \in \text{Cov-rep } A$ corresponding to π , there exists a ρ in $\text{Irr } A$ such that $\text{Supp } \mu_L = (G \cdot \rho^{-1}(0))^-$. Then, it follows that $L^{-1}(0) = \bar{\rho}^{-1}(0)$. By Lemma 3.4, one has $(\text{Ind } \rho)^{-1}(0) = C^*(\bar{\rho}^{-1}(0); \alpha)$. Therefore, one obtains that $(\text{Ind } \rho)^{-1}(0) \subset \pi^{-1}(0)$, which means that $\pi^{-1}(0) \in \{(\text{Ind } \rho)^{-1}(0)\}^-$.

Summing up the argument, we get the following:

COROLLARY 3.6. *Let (A, G, α) be as the preceding. Then $(\text{Prim } A/\alpha)^\sim$ is dense in $\text{Prim } C^*(A; \alpha)$ and*

$$\text{Prim } C^*(A; \alpha) = \bigcup \{ \mathfrak{P}^- : \mathfrak{P} \in (\text{Prim } A/\alpha)^\sim \}.$$

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